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Nonlinear nonlocal Cauchy problems in Banach spaces

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Abstract

The aim of this paper is to study the existence of integral solutions for an abstract nonlinear Cauchy problem with nonlocal initial conditions. The approach relies on the use of the theory of nonlinear semigroups and Schauder's fixed-point theorem.

Keywords: Nonlocal Cauchy problem; m-Accretive operator; Compact semigroup

1. Introduction

In this paper we are concerned with the existence of an integral solution for a nonlinear Cauchy problem with nonlocal initial conditions of the general form:

$$\begin{cases} u'(t) + Au(t) \ni f(t, u(t)), & t \in [0, T] \\ u(0) = g(u), \end{cases} \tag{1.1}$$

in a real Banach space X. Here, $A : D(A) \subset X \to 2^X$ is a nonlinear m-accretive operator such that $-A$ generates a compact semigroup $S(t)$ $(t > 0)$, $f : [0, T] \times X \to X$ and $g : C([0, T]; X) \to \overline{D(A)}$. The study of abstract nonlocal semilinear initial-value problems was initiated by Byszewski [1,2] and subsequently has been investigated in many papers [3–7]. The motivation for these studies is that nonlocal Cauchy problems have better effects in applications than the traditional Cauchy problem with an initial value of the type $u(0) = u_0$. The present work is a nonlinear version of a result by Liang et al. [5].

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There, $-A$ is assumed to be an unbounded linear operator which generates a compact operator semigroup for $t > 0$, f satisfies a Lipschitz condition in u, while g is completely determined on $[\delta, T]$ for some small $\delta > 0$ (cf. condition (H_3) below.) In this paper, we extend the result of [5] to the fully nonlinear case, by using the theory of differential equations governed by m-accretive operators in Banach spaces, compactness methods and fixed-point techniques.

2. Preliminaries

For further background and details of this section we refer the reader to Barbu [8]. Let X be a real Banach space of norm $\|\cdot\|$. A set-valued operator A in X with domain $D(A)$ and range $R(A)$ is said to be accretive if $\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$, for all $\lambda > 0$ and $y_i \in Ax_i$, $i = 1, 2$. A is called m-accretive if it is accretive and $R(I + \lambda A) = X$, for all $\lambda > 0$. (Here, I stands for the identity on X.) In the case when X is a Hilbert space, m-accretivity is equivalent to maximal monotonicity.

Next consider the Cauchy problem:

$$\begin{cases} u'(t) + Au(t) \ni f(t), & t \in [0, T] \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where A is m-accretive in X, $f \in L^1(0, T; X)$ and $u_0 \in \overline{D(A)}$. It is well-known that (2.1) has a unique integral solution $u \in C([0, T]; \overline{D(A)})$ (see, e.g., [8, p. 124]). If u and v are integral solutions of $u' + Au \ni f$ and $v' + Av \ni g$ respectively, with $f, g \in L^1(0, T; X)$ then

$$\|u(t) - v(t)\| \leq \|u(0) - v(0)\| + \int_0^t \|f(s) - g(s)\| \mathrm{d}s, \qquad \forall t \in [0, T]. \tag{2.2}$$

The following two theorems play a key role in the proof of our main result.

Theorem 2.1 (*Ascoli's Theorem*). *Let $\mathcal{F} \subset C([a, b]; X)$ satisfy:*

(i) *For any $t \in [a, b]$, $\{f(t) : f \in \mathcal{F}\}$ is relatively compact in X;*
(ii) *$\mathcal{F}$ is equicontinuous on $[a, b]$, that is, for any $\varepsilon > 0$ and any $t \in [a, b]$, there exists $\delta > 0$ such that $\|f(t) - f(s)\| < \varepsilon$, for any $s \in [a, b]$ satisfying $|t - s| < \delta$, and all $f \in \mathcal{F}$.*

Then $\mathcal{F}$ is relatively compact.

Theorem 2.2 (*Schauder's Fixed-point Theorem*). *Let C be a nonempty bounded convex closed subset in X. If $F : C \to C$ is continuous and $F(C)$ is relatively compact, then F has at least one fixed point.*

3. Main results

Let $(X, \|\cdot\|)$ be a real Banach space, and let A be an m-accretive operator in X with domain $D(A)$. It is well-known ([8]) that $-A$ generates a nonlinear contraction semigroup on $\overline{D(A)}$, which will be denoted by $S(t)(t \geq 0)$. Let r, T be finite positive constants, and set $B_r := \{x \in X : \|x\| \leq r\}$, and $K_r := \{\phi \in C([0, T]; X) : \phi(t) \in B_r, \forall t \in [0, T]\}$.

Now, we are able to formulate our main result:

Theorem 3.1. *Assume that:*

(H_1) *$S(t)$ is compact for all $t > 0$;*

(H_2) *$f : [0, T] \times X \to X$ is continuous in t on $[0, T]$ and there exists a constant $L(r) > 0$ such that $\|f(t, u) - f(t, v)\| \le L(r)\|u - v\|, \forall t \in [0, T], \forall u, v \in B_r$;*

(H_3) *$g : C([0, T]; X) \to \overline{D(A)}$ is a continuous mapping which maps K_r into a bounded set, and there is a $\delta = \delta(r) \in (0, T)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in K_r$ with $\phi(s) = \psi(s), s \in [\delta, T]$;*

(H_4) *$T \sup_{t\in[0,T],\phi\in K_r} \|f(t, \phi(t))\| + \sup_{\phi\in K_r} \|S(t)g(\phi)\| \le r$.*

Then (1.1) *has at least one integral solution.*

Remark 3.2. Condition (H_4) makes sense, since $\|f(t, \phi(t))\|$ is bounded for $t \in [0, T]$ and $\phi \in K_r$ (cf. (H_2)), and

$$\begin{aligned}\|S(t)g(\phi)\| &\le \|S(t)g(\phi) - S(t)x_0\| + \|S(t)x_0\| \\ &\le \|g(\phi) - x_0\| + \|S(t)x_0\| \\ &\le \sup_{\phi\in K_r} \|g(\phi)\| + \|x_0\| + \max_{t\in[0,T]} \|S(t)x_0\|, \text{ for all } \phi \in K_r, \text{and some } x_0 \in D(A).\end{aligned}$$

If $0 \in A0$, one can take $x_0 = 0$ and observe that $S(t)0 = 0$ in this case, so that (H_4) would be satisfied provided that

$$T \sup_{t\in[0,T],\phi\in K_r} \|f(t, \phi(t))\| + \sup_{\phi\in K_r} \|g(\phi)\| \le r. \tag{3.1}$$

Proof of Theorem 3.1. We will divide the proof into two steps.

First, set $K_r(\delta) = \{\phi \in C([\delta, T]; X) : \phi(t) \in B_r, \forall t \in [\delta, T]\}$. For a fixed $v \in K_r(\delta)$, we define a mapping $\mathcal{F}_v$ on K_r by $(\mathcal{F}_v\phi)(t) = u_\phi$ where u_ϕ is the unique integral solution of

$$\begin{cases} u'_\phi(t) + Au_\phi(t) \ni f(t, \phi(t)), & t \in [0, T] \\ u_\phi(0) = g(\tilde{v}), \end{cases} \tag{P_1}$$

with

$$\tilde{v}(t) = \begin{cases} v(t) & \text{if } t \in [\delta, T] \\ v(\delta) & \text{if } t \in [0, \delta]. \end{cases}$$

1st step: $\mathcal{F}_v$ maps K_r into itself.

Indeed, from the definition of $\mathcal{F}_v$ and (2.2), we obtain

$$\begin{aligned}\|(\mathcal{F}_v\phi)(t) - S(t)g(\tilde{v})(t)\| &\le \int_0^t \|f(s, \phi(s))\|\mathrm{d}s \\ &\le T \sup_{t\in[0,T],\phi\in K_r} \|f(t, \phi(t))\|, \qquad \forall t \in [0, T], \phi \in K_r.\end{aligned}$$

Combining the above inequality with condition (H_4), we get

$$\begin{aligned}\|(\mathcal{F}_v\phi)(t)\| &\le \|(\mathcal{F}_v\phi)(t) - S(t)g(\tilde{v})(t)\| + \|S(t)g(\tilde{v})(t)\| \\ &\le T \sup_{t\in[0,T],\phi\in K_r} \|f(t, \phi(t))\| + \sup_{\phi\in K_r} \|S(t)g(\phi)\| \\ &\le r, \forall t \in [0, T], \phi \in K_r.\end{aligned}$$

This implies that $\mathcal{F}_v K_r \subset K_r$.

From (2.2) and condition (H_2), it follows that

$$\|(\mathcal{F}_v\phi)(t) - (\mathcal{F}_v\psi)(t)\| \le tL(r) \max_{s\in[0,t]} \|\phi(s) - \psi(s)\|, \qquad \forall t \in [0, T], \phi, \psi \in K_r.$$

Moreover, we deduce inductively that for $n \in N$,

$$\|(\mathcal{F}_v^n\phi)(t) - (\mathcal{F}_v^n\psi)(t)\| \le \frac{(tL(r))^n}{n!} \max_{s\in[0,t]} \|\phi(s) - \psi(s)\|, \qquad \forall t \in [0, T], \phi, \psi \in K_r.$$

Hence, we infer that for n large enough, the mapping $\mathcal{F}_v^n$ is a strict contraction. Thus, by the Banach Contraction Mapping Principle, $\mathcal{F}_v$ has a unique fixed point $\phi_v \in K_r$, which is the integral solution of

$$\begin{cases} \phi_v'(t) + A\phi_v(t) \ni f(t, \phi_v(t)), & t \in [0, T] \\ \phi_v(0) = g(\tilde{v}). \end{cases} \tag{P_2}$$

2nd step: Problem (1.1) has at least one integral solution.

We define a mapping $\mathcal{G}$ from $K_r(\delta)$ into itself by $(\mathcal{G}v)(t) = \phi_v(t), t \in [\delta, T]$, where ϕ_v satisfies (P_2). From the definition of $\mathcal{G}$, (2.2) and (H_2), we get

$$\begin{aligned} &\|(\mathcal{G}v_1)(t) - (\mathcal{G}v_2)(t)\| = \|\phi_{v_1}(t) - \phi_{v_2}(t)\| \\ &\quad\le \|\phi_{v_1}(0) - \phi_{v_2}(0)\| + \int_0^t \|f(s, \phi_{v_1}(s)) - f(s, \phi_{v_2}(s))\|\mathrm{d}s \\ &\quad\le \|g(\tilde{v}_1) - g(\tilde{v}_2)\| + L(r)\int_0^t \|\phi_{v_1}(s) - \phi_{v_2}(s)\|\mathrm{d}s, \qquad \forall t \in [0, T]. \end{aligned}$$

Using Gronwall's inequality, we conclude that

$$\sup_{t\in[0,T]} \|\phi_{v_1}(t) - \phi_{v_2}(t)\| \le e^{TL(r)}\|g(\tilde{v}_1) - g(\tilde{v}_2)\|.$$

Consequently, $\mathcal{G}$ is a continuous mapping on $K_r(\delta)$, since g is continuous by (H_3).

We next adapt some of the arguments of [9].

Let $t \in [\delta, T]$ be fixed, and $0 < \varepsilon < t$. Define

$$v_\varepsilon(s) = S(s - t + \varepsilon)\phi_v(t - \varepsilon), \qquad \forall s \in [t - \varepsilon, T]. \tag{3.2}$$

Invoking (2.2), we derive

$$\|\mathcal{G}v(s) - v_\varepsilon(s)\| = \|\phi_v(s) - v_\varepsilon(s)\| \le \int_{s-\varepsilon}^s \|f(\tau, \phi_v(\tau))\|\mathrm{d}\tau \le M\varepsilon, \qquad \forall s \in [\delta, T], \tag{3.3}$$

where $M = \sup_{t\in[0,T],\phi\in K_r} \|f(t, \phi(t))\|$. Since $v_\varepsilon(t) = S(\varepsilon)\phi_v(t - \varepsilon)$, and $S(\varepsilon)$ is compact while $\phi_v \in K_r$, it follows that the set $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is relatively compact in X. Then (3.3) implies that the set $\{\mathcal{G}v(t) : v \in K_r(\delta)\}$ is relatively compact in X, as well.

Next, let us examine the equicontinuity of $\{\mathcal{G}v(t) : v \in K_r(\delta)\}$ on $[\delta, T]$. Since $S(t)$ is compact for $t > 0$, $\{S(t)u : u \in B\}$ is equicontinuous for all $t > 0$, and any bounded $B \subset \overline{D(A)}$. Let $t \in [\delta, T]$ and $\varepsilon \in (0, t)$. Taking into account (3.2), we infer that $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is equicontinuous at t. Therefore there exists $\gamma(t, \varepsilon) > 0$ such that

$$\|v_\varepsilon(s) - v_\varepsilon(t)\| \le M\varepsilon \tag{3.4}$$

for any $s \in [\delta, T]$ with $|s - t| \le \gamma(t, \varepsilon), \forall v \in K_r(\delta)$. Furthermore,

$$\|\mathcal{G}v(s) - \mathcal{G}v(t)\| \le \|\mathcal{G}v(s) - v_\varepsilon(s)\| + \|v_\varepsilon(s) - v_\varepsilon(t)\| + \|v_\varepsilon(t) - \mathcal{G}v(t)\|. \tag{3.5}$$

Combining (3.3)–(3.5), we get $\|\mathcal{G}v(s) - \mathcal{G}v(t)\| \le 3M\varepsilon$, for any $s \in [\delta, T]$ with $|s-t| \le \gamma(t, \varepsilon), \forall v \in K_r(\delta)$. Thus $\{\mathcal{G}v(\cdot) : v \in K_r(\delta)\}$ is equicontinuous on $[\delta, T]$. By Theorem 2.1, $\mathcal{G}(K_r(\delta))$ is relatively compact in $C([\delta, T]; X)$.

We can now apply Theorem 2.2 to conclude that $\mathcal{G}$ has at least one fixed point $v_* \in K_r(\delta)$. Let $u = \phi_{v_*}$. Then u is an integral solution of

$$\begin{cases} u'(t) + Au(t) \ni f(t, u(t)), & t \in [0, T] \\ u(0) = g(\tilde{v}_*). \end{cases}$$

Inasmuch as $v_*(t) = (\mathcal{G}v_*)(t) = \phi_{v_*}(t) = u(t), \forall t \in [\delta, T]$, we obtain $g(\tilde{v}_*) = g(u)$ by (H_3). This implies that u is an integral solution of (1.1), and completes the proof of Theorem 3.1. □

In some mathematical models, the function g takes the form

$$g(u) = \sum_{i=1}^{p} c_i u(t_i) \tag{3.6}$$

where c_i are given constants, and $0 < t_1 < t_2 < \cdots < t_p \le T$. For instance, a more realistic model for diffusion of a small amount of gas in a transparent tube involves an initial condition of the form $u(0) = g(u)$, with g given by (3.6). This allows measurements to be made at $t = 0, t_1, t_2, \ldots, t_p$ rather than just at $t = 0$. We remark that in this case, g obviously satisfies (H_3) with $\delta = t_1$, provided that $\overline{D(A)} = X$.

The following corollaries are direct consequences of Theorem 3.1 (see also (3.1)).

Corollary 3.3. *Let A be an m-accretive operator in X with $\overline{D(A)} = X$ and $A0 \ni 0$, such that (H_1) holds. Let f satisfy (H_2) and g be given by (3.6). If also the condition*

$$T \sup_{t\in[0,T],\phi\in K_r} \| f(t, \phi(t))\| + \sum_{i=1}^{p} |c_i| \le r$$

is satisfied, then the nonlinear Cauchy problem

$$\begin{cases} u'(t) + Au(t) \ni f(t, u(t)), & t \in [0, T] \\ u(0) = \sum_{i=1}^{p} c_i u(t_i) \end{cases}$$

has at least one integral solution $u \in C([0, T]; X)$.

Corollary 3.4. *Assume that the conditions (H_1), (H_2) and (H_3) of Theorem 3.1 hold for each $r > 0$. If*

$$\frac{\| f(t, u)\|}{\|u\|} \to 0, \text{ as } \|u\| \to \infty, \text{ uniformly in } t, \tag{3.7}$$

$$\frac{\|g(\varphi)\|}{\|\varphi\|_{C([0,T];X)}} \to 0, \text{ as } \|\varphi\|_{C([0,T];X)} \to \infty, \tag{3.8}$$

then (1.1) has at least one integral solution in $C([0, T]; X)$.

Remark 3.5. Conditions (3.7) and (3.8) are satisfied if there are constants $C_1 > 0, C_2 > 0$, and $\alpha_1, \alpha_2 \in [0, 1)$ such that

$$\| f(t, u)\| \le C_1(1 + \|u\|)^{\alpha_1}, u \in X, \tag{3.9}$$

$$\|g(\varphi)\| \le C_2(1 + \|\varphi\|_{C([0,T];X)})^{\alpha_2}, \varphi \in C([0, T]; X). \tag{3.10}$$

Remark 3.6. As compared to Theorem 3.5 in [4], we no longer require the compactness of g. This enables us to cover the case when g is given by (3.6). On the other hand, we now have to impose a Lipschitz condition on f.

4. Examples

Example 4.1. Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary Γ and let $\beta : D(\beta) \subseteq \mathbb{R} \to 2^{\mathbb{R}}$ be m-accretive with $0 \in \beta(0)$. Let $p \in [2, \infty)$ and $\lambda > 0$ be given. For each $u \in W^{1,p}(\Omega)$, we define the pseudo-Laplacian operator by

$$\Delta_p^\lambda u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - \lambda u |u|^{p-2},$$

where the partial derivatives are taken in the sense of distributions over Ω. Let $L_p^\lambda : D(L_p^\lambda) \subseteq L^2(\Omega) \to L^2(\Omega)$ be given by

$$L_p^\lambda u = -\Delta_p^\lambda u, \qquad \forall u \in D(L_p^\lambda), \tag{4.1}$$

where $D(L_p^\lambda) = \{u \in W^{1,p}(\Omega) : \Delta_p^\lambda u \in L^2(\Omega) \text{ and } -(\partial u/\partial \nu_p)(x) \in \beta(u(x)) \text{ a.e. on } \Gamma\}$. Here

$$\frac{\partial u}{\partial \nu_p} = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \cos(\boldsymbol{n} \cdot \boldsymbol{e}_i), \tag{4.2}$$

where $\boldsymbol{n}$ is the outward unit normal to Γ and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n$.

It is well-known that L_p^λ is m-accretive in $L^2(\Omega)$ and $-L_p^\lambda$ generates a contraction semigroup on $\overline{D(L_p^\lambda)} = L^2(\Omega)$ such that (H_1) holds; see, e.g., [10,11].

Consider the nonlocal initial-boundary value problem:

$$u_t(t, x) - \Delta_p^\lambda u(t, x) \ni c_0 \sin(u(x)), \qquad \text{a.e. on } (0, T) \times \Omega, \tag{4.3}$$

$$u(0, x) = \sum_{i=1}^{p} c_i \sqrt[3]{u(t_i, x)}, \qquad \text{a.e. on } \Omega, \tag{4.4}$$

$$-\frac{\partial u}{\partial \nu_p} \in \beta(u(t, x)), \qquad \text{a.e. on } (0, T) \times \Gamma, \tag{4.5}$$

where $0 < t_1 < t_2 < \cdots < t_p \leq T$ and $c_i (i = 0, 1, \ldots, p)$ are constants.

Let $X = L^2(\Omega)$ and $A = L_p^\lambda$, where L_p^λ is defined by (4.1) and (4.2). Define $f : [0, T] \times X \to X$ by $f(t, u)(x) = c_0 \sin(u(x))$, for all $t \in [0, T]$ and almost all $x \in \Omega$ (where $u \in X$), and $g : C([0, T]; X) \to X$ by $g(u)(x) = \sum_{i=1}^{p} c_i \sqrt[3]{u(t_i, x)}$, for any $u \in X$, and almost all $x \in \Omega$. It is easily seen that with these choices, all of the assumptions of Theorem 3.1 are satisfied. In particular, note that (3.7) and (3.10) hold. Applying Theorem 3.1 (cf. also Corollary 3.4), we conclude that the problem (4.3)–(4.5) has at least one integral solution $u \in C([0, T]; L^2(\Omega))$.

Remark 4.2. We may replace the initial condition (4.4) by

$$u(0, x) = \int_{\delta}^{T} h(s) \log(1 + |u(s, x)|) \mathrm{d}s, \text{ for all } u \in C([0, T]; L^2(\Omega)), \text{ and almost all } x \in \Omega, \tag{4.6}$$

where $\delta \in (0, T)$ and $h \in L^2(0, T; \mathbb{R})$. If we now define $g : C([0, T]; X) \to X$ by $g(u)(x) :=$ right-hand side of (4.6), we can easily verify (3.8). Moreover, (H_3) is clearly satisfied in this case. As a result, we derive the existence of an integral solution $u \in C([0, T]; L^2(\Omega))$ for the problem (4.3), (4.6), (4.5), as well.

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